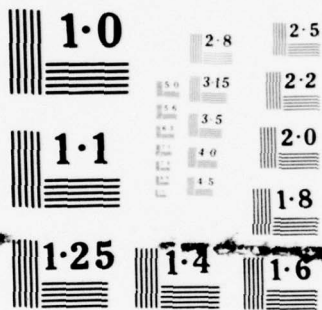


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Dimensionality and Spatial Modelling:  
A Critical Assessment\*

by

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April 1974  
Research Paper No. 22

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\* Prepared for delivery at the 1974 Annual Meeting of the Midwest Political Science Association. The research described in this paper was supported in part by ARPA Contract DAHC 15 73-C-0197.

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## Dimensionality and Spatial Modelling:

### A Critical Assessment

#### I. Introduction.

One important concern for any analysis of group decision-making is the general problem of constructing a procedure for passing from a set of known individual preference profiles to a pattern of social preferences subject to the fulfillment of certain specified conditions. In a classic study of the above problem, Arrow in Social Choice and Individual Values,<sup>1</sup> proposes certain conditions which specify desirable, while at the same time seemingly innocuous properties which every social preference ranking should satisfy. When the conditions are applied over individual preferences, the social choice is determined; but the conditions are found to be inconsistent so that no method of social choice can possibly satisfy all of the specified conditions. The social choice is shown to be either imposed or dictated. Arrow's proof demonstrates that if certain of his conditions are satisfied, the paradox of voting cannot be avoided so that given a set of transitive individual preferences, there does not result a transitive social preference.

The possibility that the paradox of voting exists such that social choices may be intransitive suggests serious problems for decision-making under majority rule, if one feels that social choice should be dependent upon the preferences of individuals in society. One approach which attempts to deal with the problem is classified under the heading "spatial models of party competition."<sup>2</sup> Generally, the spatial models approach seeks to identify, elucidate and analyze the conditions, necessary and/or sufficient, which would indicate the existence of a dominant position or equilibrium point which a candidate could choose in order to secure at least a tie in an election or a positive plurality if an opponent should choose any position which is not dominant. If certain of Arrow's conditions were modified by specifying other necessary and sufficient conditions which guarantee an equilibrium point in an election for a candidate, then Arrow's General Possibility Theorem might be avoided.



In the spatial approach, Arrow's conditions for rational social choice appear to be modified in two general areas: (1) the assumption of dimensionality in a multidimensional world and (2) the assumption that individuals act so as to maximize utility. Most of the additional assumptions in the spatial model, but not found in Arrow's work, are related in one way or another to these two assumptions. In this analysis the main concern will be with the assumption of dimensionality and several related assumptions; but it will also be necessary to include some mention of utility, since the two assumptions must be considered in concert in order to make sense of the spatial model.

The assumption of dimensionality is interesting in that the spatial model deals with the property of dimensionality in a world which is completely uninterpreted empirically, but which nevertheless has a highly developed formal or mathematical structure requiring the specific properties of continuity, infinity and single-peakedness over a set of alternatives ordered on a single dimension.

Using Arrow's work as a standard for the problem of rational social choice, it seems appropriate to ask of the spatial approach first, what are the major properties or characteristics of the spatial model? Second, is there an analogue in some other modelling enterprise from which inferences about the spatial model can be drawn? Third, do the properties of continuity, infinity and single-peakedness allow for a wide variety of possible, desirable qualities that a rational choice theory should satisfy? And fourth, returning to Arrow's formulation, how does the spatial model in general compare with Arrow's solution?

Since both Arrow and spatial analysts rely in part upon formal, empirical and theoretical assumptions in theory construction, it would seem appropriate to examine each along these lines. One acceptable criteria for theory evaluation<sup>3</sup> is: first, the theory must be examined for internal consistency by means of logical comparisons among the conclusions derived; second, the

theory must be tested for compatability with existing empirical findings or opportunities for empirical testing created by the theory; and third, the theory must be compared with competing alternatives so as to ascertain whether or not a scientific advancement has occurred.

For the most part, Arrow's formulation, demonstration and conclusions concerning the theory of social choice will be assumed as given. The reader is directed to Arrow's work Social Choice and Individual Values<sup>4</sup> for a complete presentation of his analysis. The mathematical notation concerning the problem of social choice will be based upon Arrow's logical formulation. The specific assumptions required by spatial analysis will be presented in Section II. For a more complete and detailed explication of these assumptions, the reader is directed to Riker and Ordeshook, An Introduction to Positive Political Theory.<sup>5</sup>

## II. Spatial analysis: the basic model.

According to Riker and Ordeshook, a conceptualization of a citizens's most preferred candidate is best represented by a multidimensional model such that a candidate consists of a unique position on each of  $n$  finite dimensions given as a vector,  $x = (x_1, x_2, \dots, x_n)$ , where  $x_i$  is the position a citizen most prefers on dimension  $i$ . In order to compare a citizens's most preferred position with a citizens's actual perception of a candidate on each dimension, a candidate's position may also be given as a vector,  $\theta_j = (\theta_j^1, \theta_j^2, \dots, \theta_j^n)$ , where  $\theta_j$  represents an estimate of candidate  $j$ 's position on each dimension. Thus far, the analysis assumes that each dimension relevant to a citizen's vote is representable in spatial terms. Also, the spatail analysis is not sensitive to the number of relevant dimensions and their labels.

Given the vectors  $x_i$  and  $\theta_j$  which summarize a citizen's preferences and perceptions, spatial analysis attempts to represent the utility a citizen expects to attain from  $\theta_j$ , if a citizen prefers  $x$ . The utility function relating these two vectors is given as  $U(x, \theta_j)$ . Two properties are defined in terms of the above formulation: (1) if  $\theta_j = x$ , then  $U(x, \theta_j) = \lambda$ , where  $\lambda$  is

some maximum value; and (2) if  $\theta_j \neq x$ , then  $U(x, \theta_j) < \lambda$ . Of course, an infinite but countable, number of mathematical formulations of  $U$  satisfy the two properties above. This lack of specificity with regard to the mathematical structure allows for the inclusion of several assumptions about utility functions.

One general assumption which satisfies the two properties above is:  $U(x, \theta_j)$  is concave in  $\theta_j$  so that the peak or maximum value of a concave utility function is given at  $\theta = x$  and the points to either side slope downward from  $x$ . This assumption implies a restriction equivalent to the property of single-peakedness, since the individual orderings may be represented on a graph such that the y-axis gives the rank order of the preference and the x-axis gives the set of alternatives with the result that any preference curve has one and only one dominant point or peak.<sup>6</sup> This utility function also imposes the additional requirement that the alternatives along each dimension be infinite and continuous.

The class of concave utility functions may be narrowed somewhat if only the quadratic form of the function is considered. The form is referred to as quadratic, since for one dimension the distance between  $x_1 - \theta_j$  is measured as the squared length between both positions. This length may also be treated as a norm. The more general expression of the above may be given as  $U(x, \theta_j) = \lambda - \|x - \theta_j\|^2 / \lambda$ . This expression was derived from the equation  $U(x, \theta_j) = \lambda - \sum_{m=1}^n \sum_{k=1}^n a_{mk} (x_m - \theta_{jm})(x_k - \theta_{jk})$ , where  $a_{mk}$  is the weighted sum and interaction between dimensions. It must be noted here that the magnitudes of each dimension depend upon the units of measurement for each dimension. Also the relative weights and possible dissimilar scales for each dimension and between each dimension are unknown. Therefore, the analysis is limited to theorems which are insensitive to the magnitude of each dimension. The quadratic form, in addition, indicates that  $U(x, \theta_j)$  must be symmetric about  $x$ .

Together, concavity and its quadratic form imply that as the distance between the ideal position preferred by a citizen and the perceived position of the candidate increases, utility



decreases marginally, so that the slowest rate of decrease is experienced when  $\theta_j$  and  $x$  are near one another and the most rapid when they are apart.

With the addition of the assumption of quasi-concavity, given as  $U(x, \theta_j) = \phi(\lambda - \|x - \theta_j\|_\lambda^2)$ , where  $\phi$  is any continuous monotonically increasing function, the situation in which  $U(x, \theta_j)$  decreases at a slow rate when  $x$  is far from  $\theta_j$  may also be accommodated within the analysis.

If either the quadratic form or the quasi-concavity assumption are required by the spatial model, then the following restrictions are introduced into the analysis: (1) citizens may prefer different policies, but the functional forms of their utility functions are identical; (2) all citizens weight the issues in an identical fashion; (3) citizens assign the same degree of relative importance to all issues vis-a-vis one another; and (4) all citizens use identical scales on each dimension.

### III. Physics models as analogues for spatial models.

Since the spatial model for a multidimensional world is uninterpreted in the sense that it does not specify: (1) the precise scale of measurement for each dimension, (2) the weights of each dimension with regard to others, (3) the relevant number of dimensions to be considered in the model and (4) the labels which each dimension will be assigned; it might be useful to examine the technique of dimensional analysis in other scientific enterprises, namely physics and economics, in order to assess the significance and implications of the multidimensional interpretation in the spatial model. In the process of examining the characteristics of other dimensional models, several questions are indicated and can be answered. First, what are dimensions? Second, how are they discovered? And third, how are they related to empirical, theoretical and formal aspects of the models in which they are found?

#### Three concepts of dimension.

When considering the concept of dimension, at least three varieties come to mind: ordinary language dimensions, geometrical dimensions and dimensions as concepts of measurement. All three

are found in or are potentially applicable to the political science enterprise, especially the spatial model approach.

The first concept of dimension, the ordinary language variety, treats the concept as being analyzable by ordinary language philosophy techniques which one might find in the works of Wittgenstein or Austin.<sup>7</sup> In this interpretation the concept may take<sup>on</sup> a variety of meanings for different individuals, as well as a variety of meanings for any given individual. Looking at the concept of dimension then, one finds several usages most of which may not be synonymous. For the most part, the meaning of the concept is context dependent. An example of one use of dimension would be when speaking of the complexity of some phenomenon, one might say that it had many dimensions for consideration; meaning that the phenomenon had several facets all of which are relevant in discussing the phenomenon. Another use of dimension might be discovered when contemplating the enormity of some object; in this usage, one might say, for example, that the size of a Boeing 747 is quite large in its dimensions with regard to some other object not necessarily another aircraft. To reiterate, the point being made here is that the ordinary usage of dimension is variable across contexts in which it occurs and its precision or explication is not necessarily highly developed, although its meaning is reasonably clear in every day discourse.

The second concept of dimension may be referred to as the geometrical concept of dimension. One technical instance<sup>8</sup> of the concept may be characterized with regard to vector spaces, where a vector space is symbolized as  $V$ . The vector space  $V$  contains a set of points- vectors on which two operations are defined: vector addition and scalar multiplication. Vectors belonging to a vector space may be classified into two classes: linearly independent and linearly dependent. A linearly dependent set of vectors occurs when each vector in the set lies in the same plane and each passes through the origin. If a collection of linearly independent vectors, which are a set of vectors not in the same plane, may be represented as a linear combination of  $n$  vectors, then these vectors are a basis for  $V$ . In this

interpretation, dimension becomes dimensions of  $V$  given as  $n$ . For example, in Euclidean  $n$ -space,  $E^n$ ,  $n$  represents the dimensions of  $E$ .

Notice that the geometrical interpretation of dimension differs from the ordinary language concept in that (1) it loses its ordinary language connotations becoming a technical term, (2) its meaning is precise so that agreement and disagreement about its properties and characteristics may be discussed in common terms, and (3) it is reasonably clear where the concept fits into the rest of mathematics. The two concepts when viewed comparatively may be seen not as correct or incorrect, but more advantageous or appropriate with regard to their use in understanding certain phenomenon.

Now, it appears that the political science notion of dimension may occur variously under both concepts. Of prime concern here would be its geometrical use, while language philosophers are concerned with the former concept. In the political science usage or application of the concept, if I understand it correctly, an interpreted mathematical structure is somehow mapped into some political phenomenon in the world or in a possible world. The procedure for mapping one structure into another may be undertaken by developing a model and mapping it into a world, or by taking some aspect of a world and mapping it into a model. Next, some numerical assignment is given to the elements of the vectors according to some rule of measurement; this is called quantification or perhaps scaling in S. S. Stevens' sense of the term.<sup>9</sup> The quantified phenomenon is manipulated by any variety of techniques, in this case vector addition and scalar multiplication. A solution is obtained from these operations and this is taken to be a "description" or "explanation" of the political phenomenon.

In practice, the above use of dimension in political science and spatial modelling leads to certain paradoxes and inconsistencies which are disturbing. In the case of spatial modelling, consider first a situation wherein the following circumstances arise. Let a dimension express the quantity: government aid to education, measured on scale  $A$ . Suppose that 50% of the voters



agree and 50% disagree on the alternatives. According to the spatial model, either position is acceptable to a candidate. Next, introduce another dimension: role of government, state versus federal, with regard to social programs measured on scale B. Suppose that 100% of the voters agree with the federal support portion and 0% agree with the state support alternative. The spatial model would indicate that an optimal location occurs on the federal government alternative given at 100% agreement. Next, introduce the dimension: aid to education as scale C. Let 100% of the voters favor the position and 0% disagree. Again the equilibrium point would be at 100% agreement. One conclusion from scales A, B, and C is that scale A really is dichotomous being composed of scales B and C, where an optimal strategy suggests locating at positions which favor federal support and aid to education. Now consider a situation in which a dimension is given as federal aid to education, measured on scale A'. Suppose that the voters split on this issue 50% agreeing and 50% disagreeing, so that either the disagree or agree position would be appropriate for a candidate location. Clearly, the above conclusion does not follow with the addition of this new scale A' since one would have expected the voters to align 100% in agreement on this issue. Indeed, this phenomenon, although perhaps not manifested as clearly as above, occurs frequently in survey data analysis. The results suggest that different scales, although relating to the same quantity or relationship, may lead to alternate solutions.

Instead of viewing the above as the consequence of an inappropriate use of the concepts dimension and scale, political science suggests that there may be an error factor creeping in which alters the results. For example, some respondents may choose always to respond positively to a question no matter what it says. Usually, then, the factor is added into the model: in regression analysis, an "e" is added to the regression equation. Another frequently observed explanation, given if the error explanation is not satisfactory, would be that respondents in a survey are somehow irrational or have undeveloped belief systems, or have low centrality among attitudinal components.

Clearly, the phenomenon cannot be explained in terms of the concept of dimension being used, but must instead <sup>resort</sup> to other kinds of explanations. Given that these kinds of phenomenon arising out of the geometrical use of dimension are undesirable, perhaps it would be useful to search for another interpretation of dimension which can account for these occurrences, even if it may not in fact solve them,

This leads us to dimension as a concept of measurement.<sup>10</sup> In this interpretation, the analyst is concerned not only with dimension in a geometrical sense, but also with respect to measurement considerations so that the interaction of both components becomes significant.

#### Dimension as a concept of measurement.

In order to understand the concept of dimension, it is necessary first to understand the concept of quantity,<sup>11</sup> symbolized as  $q$ . Generally, quantity is expressed as a magnitude multiplied by a unit of measurement. An example of the quantity "time" would be 10 seconds, where 10 is the magnitude and seconds is the unit of measurement. Although quantities are expressed in terms of magnitude and unit, quantities are independent of both (this will be demonstrated later on in the analysis). Most sciences view the concept as a primitive term in the context of justification. Quantities are classified as primary and secondary. A primary quantity is one in which the units are considered to be fundamental in the sense that they are not reducible to any other quantity. In physics, these would include mass, length and time. A secondary quantity is one which is composed of a combination of primary quantities in a functional relationship. An example would be the equation for force, where force is equal to mass times acceleration,  $f = ma$ . The designation of primary and secondary quantities is entirely arbitrary and depends for the most part upon the particular set of rules governing a scientific paradigm which are convenient to adopt in defining a system of measurement and upon the purpose of the analysis. In some systems, for example, the quantity force may be given as a primary quantity.

The scales used in measuring quantities are also important in understanding the concept of dimension.<sup>12</sup> By scale I mean (1) a rule for making numerical assignments to some phenomenon, (2) so that the same numerical assignment may be given to the same object under identical conditions and (3) so that the possibility of assigning different numbers to different things under the same conditions exists. Scales are usually given as similar and dissimilar when related to quantities. A similar set of scales suggests that all of the units used to characterize a quantity may be converted to any other unit so that an absolute ratio between two measurements remains the same regardless of a change in unit. An example for the quantity time would be 60 seconds are contained in one minute and one minute is contained in one hour. A class of dissimilar scales does not allow for conversion of one unit to another while at the same time preserving an absolute significance between two measurements. For example, time measured in seconds could not be converted to time measured in "dogs" so that the same absolute significance is preserved. This is so since dogs may be measured in terms of weight, color, volume, number, breed, etc., which may vary across the set of all dogs so that a relationship which transforms seconds into dogs is not possible.

#### Dimension defined.

Keeping in mind the explication of quantity and scale above, the concept of dimension as measurement may be defined as an expression of a quantity in terms of one class of similar scales.<sup>13</sup>

The combination of two or more quantities by means of two or more dimensional expressions of these quantities is given as a functional relationship of the general form:

$$g = f(x_1, x_2, \dots, x_n),$$

where  $g$  is a secondary quantity expressed as a secondary dimension,  $f$  is a function and  $x_1$  through  $x_n$  are dimensions of quantities in the domain of the function.<sup>14</sup>

#### How dimensions are discovered.

Having defined dimensions as a concept of measurement in terms of quantity and scale, the analysis will turn next to an



explanation of how dimensions are discovered. A practical example which might be cited would be the determination of the time of swing of a simple pendulum. Potential quantities to be considered might be:

<u>name of quantity</u>	<u>dimensions</u>
time of swing	t
length of pendulum	l
mass of pendulum	m
acceleration of gravity	g
angular amplitude of swing	$\theta$

By combining the above in the most general functional form, one obtains  $t = f(l, m, g, \theta)$ .<sup>15</sup> Clearly, all of the above dimensions in the domain of the function seem completely plausible and potentially relevant, but which ones are relevant and what is the specific form of the function combining them? In other words, how is the correct equation,  $t = f(\theta) l/g$ , which has been determined by dimensional analysis in physics discovered?

One strategy for solving the above problem would be to combine all of the above dimensions in a multitude of ways concerning every possible combination, and then test each empirically to discover an appropriate formula. Certainly this is infeasible, first, because as the number of dimensions increases the number of combinations increases also so that empirical confirmation becomes increasingly more difficult or even impossible; and second, because there is no guarantee that all of the relevant dimensions are in the list to be analyzed.

Another strategy would be to formulate a mathematical or formal structure prior to the determination of either the quantities or dimensions in the hope that the correct structure has been chosen. Of course, the strategy evidences at least two major problems: first, there are an infinite number of structures which may or may not accommodate the relevant dimensions, and second, there is no guarantee that all of the relevant dimensions may be accounted for in the structure which is being posited.

The most plausible explanation for the discovery and manipulation of dimensions- an explanation which will point up

further difficulties with the two strategies mentioned above- is one which elucidates the formal relationship between quantities dimensions and numerical laws. To begin, numerical laws may be defined as a functional relationship between two or more quantities under specified conditions which may be confronted with data. Numerical laws are expressed in functional form in a manner identical to the dimensional formulae except that the numerical law is independent of the dimensions which may define it.<sup>16</sup>

Since the relationship between quantities is defined as constituting a numerical law, then if quantities are independent, numerical laws must be independent also. This may be shown in three ways. First, take a quantity "time" as the phenomenon for analysis. In order to demonstrate that time is independent of the dimensions which may define it, consider at least two classes of scales which are independent of one another and are used to measure time: clock time and mathematical time. Clock time, expressed in seconds, is simply our everyday means of indicating time. One notion of mathematical time would be an attempt to characterize time as being a geometrical entity having length or extension. Not only does mathematical time differ in that it has extension, but within the quantity one finds that the kind of distance function, that is, Euclidean and non-Euclidean, offers an infinite number of possible dimensions, none of which reduce to time in seconds, minutes or hours in the sense of being classes of similar scales. Clearly, if the same quantity may be expressed according to a wide variety and infinite number of independent scales, yet still refer to the same phenomenon, then the quantity must exist independently of the dimensions defining it.

Second suppose that an analyst desires to measure the quantity gravity, given as  $g$ . Through empirical testing and deductive manipulation, suppose that the secondary quantity for gravity is determined to be  $m^{-1}w = g$ , where  $w$  is the quantity weight and  $m^{-1}$  is the reciprocal of the quantity mass. Another analyst sees this and proposes a competing formula  $g = lt^{-2}$ , where  $l$  is the height an object falls and  $t^{-2}$  is the square of

the reciprocal of time during which the object falls. Both formulae are measuring the quantity gravity, yet each is completely independent and not derivable from the other in terms of the paradigm governing measurement. From this one can see that the secondary quantity gravity must be independent of the dimensions of the primary quantities which define it. And, the primary quantities are also independent of the dimensions which define them. Therefore, since a numerical law is composed of quantities, it too must be independent of the dimensions which define it.

And third, since the designation of primary and secondary quantities and their expression in terms of dimensions is entirely arbitrary, depending upon the purpose and (as will be discussed later) the theoretical framework of the analysis; it is not impossible for any quantity or collection of quantities to have the same dimensions. In spite of this, the quantities and numerical laws remain the same. Therefore, again the independence of quantities from dimensions which express them is indicated.

Several formal implications of dimensions as viewed above.

Points one through three above, suggest several important consequences for the use of dimensions in an analysis.<sup>17</sup> One would be that even though quantities and numerical laws are independent of the dimensions which may be used to express them and even though the specification of primary and secondary quantities is for the most part arbitrary, the choices in an analysis which manifest the character of laws and quantities determine, in part, the functional form the dimensional equation can assume. In like manner the choices with regard to classes of similar scales in which dimensional equations are represented also determine in part the formal nature of the dimensional equation. Both opportunities requiring choices, then, may be seen as limiting the formal, mathematical structures of dimensional equations.

Another consequence would be that again, since quantities are arbitrary, and quantities may be expressed in terms of



different laws and classes of similar scales, solutions to problems concerning the same phenomenon will necessarily vary with the possible result that they are contradictory or irrelevant with regard to one another or no comparison may even be possible. Therefore, in order to make sense of the solution to a problem, it is necessary that the prior determinants be specified in order to discover which interpretations are contradictory, irrelevant or indeterminate. If the prior determinants are not specified, then the dimensional equation provides a solution, but it will not be possible to decide for which problem it happens to be a solution. If solutions are contradictory, depending upon prior choices, then clearly the problem is highly significant, because an analyst does not know which solution to accept as appropriate.

Yet another consequence would be that: if the specification of quantities is arbitrary, if a single quantity may be expressed in terms of alternative numerical laws and if by definition classes of similar scales are independent of other classes of similar scales, then it can be shown to follow that a dimensional equation cannot be used to deduce the formal structure of a numerical law a priori. Suppose, for example, that an analyst is given an uninterpreted dimensional equation  $xy = z$ . Clearly, this equation may represent any numerical law ranging from  $f = ma$  to  $E = mc^2$ . Therefore, in general one may conclude that uninterpreted dimensional equations are of dubious value when presented devoid of the results of prior knowledge which must have preceded them or should have preceded them. The uninterpreted equation may still be of interest from a pure mathematical point of view, however. Next, consider the case wherein an interpreted model is presented a priori. Initially, it seems clear that it would be difficult or impossible to think up dimensions which do not refer to well established numerical laws. Even if the dimensions happened to be appropriate, there exists no a priori experience which would dictate their structure short of listing every structure possible or guessing about the nature of any particular one. Even if one grants that dimensions may be thought up and their structure determined, there still

appears to be no way of knowing whether all of the dimensions are included. An example of this might be that one could list all of the dimensions in the previous pendulum problem and derive its precise structure and correctly represent a law, but in order to do this one would have had to have knowledge of most all of the other laws of physics, since the dimensional equation exists with a ceteris paribus assumption with regard to these laws. In other words one would have to know that the laws of gravitational attraction, quantum mechanics, etc, should or should not be included.

A final consequence in light of the above would be that it is incorrect to consider dimensions as transformation formulae between scales.<sup>18</sup> This follows directly from the fact that dimensions are determined by choices of numerical laws and similar scales so that a dimension in one class of similar scales cannot be converted into the same dimension in another class of similar scales. Therefore, there exists quantities which cannot be expressed such that given magnitudes and units every dimensional representation may be converted into every other for any quantity which is designated as a primary or secondary quantity. Instead, it appears to be the case that the following characteristics are indicated: (1) transformation formulae express functional relationships between magnitudes of quantities which are uninterpreted, but which have similar or equivalent scales, (2) dimensional formulae express functional relationships between quantities being expressed as one class of independent, similar scales and magnitudes and (3) numerical laws express functional relationships between quantities, either primary or secondary, which are independent of dimensions which may be used to express them.

#### Spatial models: some formal implications.

In considering the spatial modelling approach in light of the formal characteristics of the analysis thus far, the following implications emerge. First, it has been noted above that our prior choices among expressions for quantities and numerical laws determines the structure of dimensional formulae. In the

spatial model, one is given a voter's vector space  $x = (x_1, x_2, \dots, x_n)$  and a candidate's vector space  $\theta_j = (\theta_j^1, \theta_j^2, \dots, \theta_j^n)$  which stand for dimension in the geometrical sense of the term. Both vectors are combined into a numerical law given as  $U(x, \theta_j)$ , where this expression equals:

$$A = \sum_{m=1}^n \sum_{k=1}^n a_{mk} (x_m - \theta_{jm})(x_k - \theta_{jk}).$$

Clearly, this expression for quantities, laws and dimensions does not in fact conform to the idea that the prior choices must act as determinants for the dimensional formula. As it now stands, the expression of the formula is true by definition since it is equivalent to the "general form" of a dimensional equation given earlier. What it does not do is specify the precise relationship which each dimension must have to every other dimension as well as to the entire set of dimensions. This of course applies here to the case of one individual and one candidate; the problem is more serious and complex when additional voters and candidates are added.

Second, the analysis has suggested that dimensions as concepts of measurement may generate problem solutions which are contradictory, irrelevant or indeterminant. In spatial modelling, consider a case wherein the same dimension has two scales which are not similar by definition, rather than derived empirically as above in the "aid to education example" and which lead to contradictory solutions. Suppose that an analyst proposes an issue dimension which has one scale given as a valence issue- "those that merely involve the linking of the parties with some condition that is positively or negatively valued by the electorate;" and the other scale as a position issue- "those that involve advocacy of government action from a set of alternatives over which a distribution of voter preferences is defined." Let the valence scale and position scale be dissimilar. Further, let each scale exist such that if one is chosen the other is precluded from use. Suppose that the issue is given as a position scale and 100% of the voters agree and 0% disagree. Next, the same issue is given as a valence scale and 100% of the voters disagree and 0% agree. Clearly, the same dimension has one



equilibrium position on one scale and another on the second. Therefore, the choice of scales may lead to different solutions. Now, let both scales be dissimilar, and stipulate that they occur concomitantly in the same dimension, but the precise nature of their interaction is indeterminant. Certainly, in this case the spatial model cannot account for the desired equilibrium point. This remains the case even though the separate scales above might be considered. Therefore, using the combined scale will lead to a solution which may be contradictory, irrelevant or indeterminant. The obvious relevance of this example for spatial modelling is that if the dimensions are not specified, but left in geometrical form, the analysis can be shown under certain circumstance to generate an equilibrium point; but the more important question is an equilibrium point in relation to what? This cannot be addressed in the model.

Third, the above analysis has indicated that although it is possible to construct dimensional formulae from quantitative numerical laws, it is not possible, except perhaps in some fortuitous manner to construct dimensional formulae a priori and attempt to deduce the structure of numerical laws from them. Yet in the spatial model being analyzed, this is precisely what is being done. In effect, the use of the most general functional form in terms of the geometrical concept of dimension is presented as if it were possible to deduce the formal structure when given dimensions as concept of measurement. In order to show that this is certainly not possible in the model, consider (1) quantities are not specified in terms of primary and secondary classifications, (2) point one immediately precludes the possibility that numerical laws are or can be specified, (3) classes of similar scales are defined, (4) the nature of the solutions are not specified, and (5) most importantly, the model itself is only presented in a very general form. Given the above impediments, it is difficult to see how the model can be used to aid our understanding of numerical laws.

#### Several empirical implications of dimensions.

Thus far the analysis has shown that the formal structure

of dimension is determined by the choice of numerical laws and similar scales, but there is also an empirical component whose necessity can be demonstrated<sub>20</sub> and the nature of which determines dimensional formulae in part.

The necessity and determinancy of the empirical component in general may be demonstrated in the following example. In mathematics, it is known by definition, derivation and deductive proof that the 1st through the nth derivative of any equation may be computed. For example, the equation  $f(x) = x^3 + x^2 + x + 6$  has a 1st derivative  $3x^2 + 2x + 1$  and a 2nd derivative  $6x + 2$  and a 3rd derivative of 6 and so on. In physics, the variables in the equation are made specific such that  $s$  = distance,  $v$  = velocity,  $a$  = acceleration and  $t$  = time. By definition and convention, distance is a function of time,  $s = f(t)$ ,  $v$  is the 1st derivative of  $s$  with respect to  $t$  and  $a$  is the 2nd derivative of  $s$  with respect to  $t$  or  $a$  is the 1st derivative of  $v$  with respect to  $t$ . These relationships constitute the basic structure for the laws of motion in physics.

Given that the laws of motion in physics are based on the above relationships, it can be shown that it is the empirical phenomenon which gives the equations their form or structure, while the uninterpreted mathematical equations provide the rules and conventions for defining and manipulating variables. For example, using the symbolic notation for equations of motion  $a = D_t v = D_t^2 s$  provides no clue as to the actual equations of motion which are derived by testing a phenomenon empirically, but merely show conventional analytic relationships. In this example, if an analyst asked how fast a rocket is traveling, then the answer is  $D_t s = v$ . However, if an analyst wished to know the precise equation which expresses the relationship so that if by observation, values for  $s$ ,  $v$ ,  $a$  and  $t$  are obtained, then one possible equation might be  $f(s) = t^2 + t + 1$  and velocity could be given as  $D_t s = v = 2t + 1$ . Clearly, mathematics provides the set of rules and the set of all possible structures, but it is the nature of the empirical phenomenon which is given interpretation in the form of an equation which makes the model apply to the phenomenon.

The empirical component is perhaps best demonstrated by the nature of dimensional constants in dimensional formulae.<sup>21</sup> A dimensional constant is a constant which has dimensions so that a change in numerical magnitude is accompanied by a change in the size of the fundamental units involved. The obvious importance of dimensional constants may be illustrated as follows. Suppose an analyst wishes to discover the gravitational attraction between two objects. All of the important dimensions and quantities may be listed for analysis:

<u>name of quantity</u>	<u>dimension</u>
mass of first body	$m_1$
mass of second body	$m_2$
distance between bodies	$r^2$
time of revolution	$t$

The most general form would be given as  $t = f(m_1, m_2, r)$ . It is clear that on the left side of the equation one finds a unit of time, but on the right, no such dimension is possible. Therefore, one might conclude that even though all of the variables which may be varied are included, there must be something missing from the equation which when included will make the functional expression true. There are of course, an infinite number of formal expressions which may be considered, but there is no a priori way of discovering the nature of this expression. In this case, the missing element would be the gravitational constant  $G$ , given as  $m^{-1}l^3t^{-2}$ . The equation then becomes  $t = G m_1 m_2 / r^2$ . The inclusion of the constant significantly alters the expression. Further, the constant is not apparent in any of the quantities which are listed as relevant, and is therefore not derivable from any of the dimensions not matter which ones would be indicated. Subsequently, one may ask, how is it that the constant  $G$  comes to be included in the dimensional formula? The constant is derived by knowledge of the empirical phenomenon and from the use of certain other numerical laws which indicate that the constant is appropriate the and indeed necessitated. This illustrates that dimensional formulae are not only formal expressions, but also highly empirical in nature.

Sometimes within a dimensional analysis, the dimensional constants may be left uninterpreted. Generally, the consequences



of doing this are as follows: if the number of dimensional constants is less than the number of variables being considered in an analysis, then dimensional analysis may proceed but solutions may include an unknown constant or constants. This is not entirely undesirable, depending on the purpose of the analysis and the nature of the solution desired. If the number of constants is equal to the number of variables in an equation, then the equation can provide no information at all and dimensional analysis is impossible.

To summarize briefly, empirical components enter into an analysis in at least three places: (1) past experience of other laws, (2) use of laws in devising dimensional formulae and (3) the inclusion of dimensional constants in dimensional formulae.<sup>22</sup>

#### Several empirical implications.

The notion that dimensional equations are by necessity part empirical has several implications for dimensional models. Each of the following points may be seen to parallel or correspond closely to the implications arising out of the formal section of this analysis.

First, even though it appears that the formal or mathematical rules and structures seem to make numerical laws and dimensional equations definitional, this is not the case since it is the interpretation of the empirical phenomenon which dictates the form of the law and structure. This was evidenced by the discussion of the equations of the laws of motion and the use of dimensional constants. Therefore, just as the formal structure of numerical laws was seen as determining the dimensional equation for the law, the empirical component when interpreted may be seen as a determinant in part of the formal structure of numerical laws. Clearly, the empirical and formal determinants of numerical laws must precede the construction of a dimensional equation. If this interpretation of dimensional analysis is not followed, then dimensional formulae are not really dimensional in the sense of the concept of measurement, but are instead geometrical dimensions which are of interest in mathematics only.

Second, the nature of numerical laws suggests that the same empirical phenomenon may be interpreted in many ways according to the prior choices made in expressing them. Further, the analysis has shown that the same mathematical structure applies to many empirical interpretations of a phenomenon, but certainly not all. If this view is correct, then it follows that in order to make sense of the solutions to an analysis, the empirical and formal determinants must be specified. If this specification is not forthcoming, contradictory, irrelevant and indeterminate solutions cannot be identified. This lack of solution identification, then, would mean that we would not know what the analysis means in relation to the phenomenon.

Third, given that the empirical and the formal aspects of numerical laws are variable across interpretations, that contradictory, irrelevant and indeterminate solutions may exist and that the empirical component is a necessary element in the analysis of a phenomenon; a conclusion that may be drawn is that an uninterpreted dimensional analysis, where dimensions are geometrical entities, does not admit of the possibility of an a priori discovery or specification of numerical laws.

Spatial models: some empirical implications.

Having detailed some of the empirical implications for dimensional models, the analysis will attempt to relate these implications directly to the spatial model.

The first implication which must be considered is that dimensional formulae are in part empirical and that this empirical portion determines the formal structure of the dimensional formulae. Consider in the spatial model the mathematical property of single-peakedness and the numerical law which suggests that individuals act in such a way as to maximize utility. Other analyses have shown that single-peakedness over a set of individual preferences will guarantee a best social choice in a model. This remains the case no matter what one calls the orderings, that is, it does not matter whether individuals are maximizing utility or acting according to some other decision criteria. Therefore, the model gives a sufficient condition for guaranteeing an equilibrium point independent of whether

the empirical phenomenon of utility maximization for individuals exists.<sup>23</sup> Consequently, it appears that the addition of the assumption of utility adds nothing to the spatial model since single-peakedness already has guaranteed the results. More importantly, the empirical nature of utility maximization as viewed dimensionally exists in name only, or by definition only, and has not really determined even in part the structure of the model as the previous analysis has suggested that it should.

By not considering the empirical component in utility maximization, and by relying only upon the logical property of single-peakedness, the spatial model may be seen as highly restrictive in several potentially undesirable ways: (1) there are a good many other formal properties which also guarantee a best social choice, but are not accounted for in the spatial model. Among these would be the qualitative properties: dichotomous, echoic and antagonistic preferences; value-restricted preferences; single-caved and two-group-separated preferences; and taboo preferences.<sup>24</sup> All of these properties may be presented in terms of utility, but they work independently of the property as well; (2) the property of single-peakedness requires that the number of individuals concerned in social choice be odd, if a best point and not an equilibrium point is desired. Clearly, this limits the model since it cannot guarantee a best point, but merely an equilibrium point point for even numbers of individuals or free numbers, that is, numbers of individuals where oddness or evenness is irrelevant. In the above alternative properties, dichotomous, echoic and antagonistic preferences provide a best social choice when the number of individuals free;<sup>25</sup> (3) it is possible to find empirical examples where single-peakedness would not apply in important decision-making contexts. Take for example roll call voting in the United Nations Assembly; clearly the property of single-peakedness as a quantity common to preference orderings is not applicable in all cases; and (4) the property of single-peakedness as it stands in the spatial model can only demonstrate a sufficient condition for an equilibrium point, but not a necessary one.<sup>26</sup> This suggests that the property is much weaker analytically than other rational choice theories which



have specified necessary conditions for an equilibrium point as well as a "best" point.

The spatial model may also be analyzed with regard to empirical implications by examining the nature and function of dimensional constants within the dimensional model. Recall the general expression:  $R = \sum_{m=1}^n \sum_{k=1}^n a_{mk} (x_m - \theta_{jm})(x_k - \theta_{jk})$ , where the dimensional constant  $a_{mk}$  is intended to represent the weighted sum and the interaction between each pair of dimensions. Clearly, from the above analysis, dimensional analysis in any empirical sense is impossible since the dimensions and dimensional constants are expressed in terms of unknowns. Therefore, the expression can provide us with no information about the phenomenon under analysis. The obvious rejoinder to this would be that all that needs to be accomplished is the empirical interpretation and analysis of the unknown expressions and a solution will be attained. But, this is precisely the criticism being offered in this analysis. The first important point to be noted here is that in spatial terms, the entire problem of social choice reduces to the precise expression of the  $a_{mk}$  constant. In essence then the expression of the entire mathematical structure is somewhat meaningless without information about  $a_{mk}$ . More importantly, the rejoinder does not consider the fact that the dimensional constants, in physics at least, cannot be in many cases discovered within the relevant dimensions of the phenomenon under study, but instead derive from other empirical analyses over other dimensions. This it will be recalled is the case in the expression of  $G$  the universal gravitational constant. If this view is correct, then the spatial model has really not solved the problem, but instead has shown that the constant must be determined in some other analysis.

Another implication of the empirical analysis was that since there are possibilities for contradictory, irrelevant and indeterminate solutions to dimensional problems, it is necessary for a dimensional model to specify the exact empirical interpretations and components so that the solutions to problems can be made sense of according to the notion of dimension as a concept of measurement. When the spatial model is considered

in light of this implication it will be necessary only to recall the example of "government aid to education" presented earlier in order to assess the consequences of empirical phenomenon for the spatial model. If the empirical components are not considered, it will be impossible for the spatial model to give a solution which can be meaningfully evaluated, since the empirical phenomenon leads to two different equilibrium points. Since one purpose of spatial modelling is to discover a unique equilibrium point, the non-empirical aspect of the model appears to be unsatisfactory.

Yet another implication involved when considering empirical aspects of a dimensional analysis is the notion that numerical laws cannot be deduced from uninterpreted mathematical structures. This conclusion implies important consequences for the analysis of empirical phenomenon by means of the spatial model. Initially, upon examining the spatial model, one finds that the numerical law for individual utility maximization over a multidimensional world is initially posited in the analysis. The law of utility maximization, although assumed, does not enjoy extensive theoretical acceptance or empirical support vis-a-vis other alternatives. Among the more prominent alternate explanations, one finds the following: (1) given the high cost of information and lack of information, individuals may seek a "satisfactory" choice, rather than some optimum one;<sup>27</sup> (2) individuals have a propensity to act out of interest in a game or gamesmanship so that even when alternatives are known, and probabilities are given, individuals attempt to beat the odds, thereby not maximizing utility;<sup>28</sup> (3) in some cases choosing one's most preferred alternative in a voting situation, may in a sense be wasting it, since it may not be a possible winner; whereas if a most preferred alternative is abandoned in favor of some less preferred alternative, then perhaps some gain may be achieved: this point suggests concepts such as "strategic" voting,<sup>29</sup> logrolling and Bayes minimax<sup>30</sup> strategies which not only concern some maximum choice, but also the notion of a minimum, as well as positions in between; (4) some individuals may not vote for utility on an individual level, but instead out of altruistic motives; (5)

it is frequently the case that individuals possess little information about politics, and vote out of an interest in citizen duty rather than utility maximization;<sup>31</sup> (6) it is not clear how utility would relate to undeveloped belief systems in which there exist no opinions and non-attitudes,<sup>32</sup> since information appears to be a necessary condition for expression of an individual's position on a set of dimensions; and (7) for a person to have an optimum choice with a subsequent ordering should not depend upon the order in which the choices are presented, and the ordering should not change when the order in which the alternatives are presented changes without some genuine attitude change, empirically, this seems for some individuals not to be the case.<sup>33</sup>

From the above presentation the analysis has suggested with regard to utility maximization, that there is substantial evidence that the phenomenon may not be especially warranted in many decision-making contexts. Combining this notion with the previous conclusions that the formal structure for utility maximization is unknown, there are a multitudinous variety of ways in which any law could be expressed, some of which may be mutually exclusive; and the nature of laws and quantities may lead to solutions which are contradictory, irrelevant or indeterminate; it seems clear that the possibility for deducing numerical laws is at worst impossible and at best extremely fortuitous.

#### Spatial modelling: can it be justified?

Thus far, the formal and empirical components of dimensional models in general, have serious implications for the spatial model approach. Perhaps the polemics in this regard may be presented as follows: if the analysis is correct in asserting that formal and empirical quantities and numerical laws must exist and must be developed in order to derive a dimensional equation and if the converse is not correct, then the spatial model cannot be developed a priori. If it cannot be developed a priori, then it cannot be used to discover laws, in this case utility maximization in a multidimensional world. Instead, it appears to remain entirely definitional. The spatial analyst



might object that although the spatial model is uninterpreted, it may at some time, using the given structure, be made interpreted and subsequently tested empirically. But, the problem with this rejoinder is evident: if one needs to interpret a structure, map it into some empirical representation of a phenomenon and test it empirically, then the spatial model appears to necessitate an additional, yet unrequired step. This may be shown by observing that if the empirical phenomenon is well enough understood to be cast in dimensional terms, it must be well enough understood to be cast in terms of quantities and numerical laws. Since dimensional analysis is an analysis of an analysis, a dimensional interpretation derives also from this empirical investigation. The extra step occurs in that this derived dimensional analysis must be compared with the posited a priori model. Clearly, if an analyst has a "properly" derived dimensional model, it would not be especially productive to have an a priori one also. Furthermore, the complexity of the simple physics problems like the pendulum problem above, suggest that the possibility of attaining a derived and an a priori model which are identical is not high.

Spatial modelling: the problem of continuity, infinity and constraints.

Thus far the analysis has discussed some of the apparent consequences- formal and empirical- when developing an uninterpreted dimensional model prior to developing its antecedents. There exists another problem of an opposite nature, when one examines the highly specific mathematical assumptions necessary to construct the dimensional structure of the spatial model. The properties of continuity, infinity and single-peakedness over alternatives on a dimension taken as assumptions may serve to illustrate this point.

One potential problem created by the formal assumptions of infinity and continuity over a set of individual preference orderings occurs when constraints are introduced into the spatial model<sup>34</sup> so that certain alternatives which may be desirable and most preferred become infeasible. A possible example of this problem would occur in the real world environment when individuals

desire that  $x$  amount of dollars be spent on a social program, but a budget constraint of  $x-1$  dollars makes the most preferred amount  $x$  infeasible. The problem above may be characterized as follows: the assumption of single-peakedness is a sufficient condition for the existence of an equilibrium point  $x$  over the alternatives in one dimension; but if the equilibrium point  $x$  is outside the range of feasible alternatives, then one question becomes, is there some unique point in each subset of ordered alternatives which represents a best alternative or equilibrium point?

The following analysis will demonstrate that the existence of an equilibrium point over any subset of alternatives, where a constraint is imposed such that  $x$  represents only a unique solution to an unconstrained problem, may not be guaranteed. To begin, the following notation and definitions will be offered.<sup>35</sup> Let  $S$  be the set of all possible alternatives to be considered for social choice. And let  $A$  be any given subset of  $S$ . Next, the concept of a maximal set may be defined; for any given subset of  $S$ , the maximal set  $M(A)$  is:

$$(\forall x) \left[ \{x \in M(A)\} \leftrightarrow (x \in A) \ \& \ (\forall y) ((y \in A \rightarrow \sim(xPy)) \right],$$

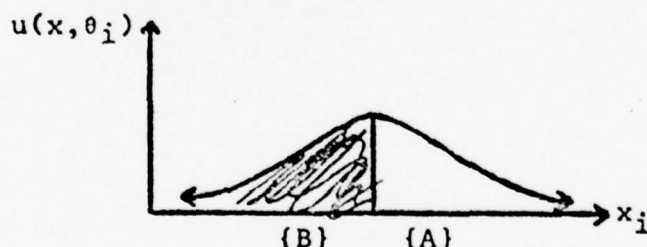
which means there exists a set of alternatives in  $A$  such that no better social alternative in  $A$  may be found. From the definition for a  $M(A)$ , the definition for a choice set  $C(A)$  is given as:  $(\forall x) \left[ \{x \in C(A)\} \leftrightarrow \{(x \in A) \ \& \ (\forall y) ((y \in A) - xRy)\} \right]$ , which means that there exists some element of  $A$  which is at least as good as any other element  $A$ .

The maximal set  $M(A)$  and the choice set  $C(A)$  may be shown to be related as follows:  $C(A) \subset M(A)$ . If the alternatives in  $S$  are both reflexive and connected, then  $C(A) = M(A)$ . Of course, a unique element in  $C(A)$  would be equivalent to an equilibrium point. Using the concept of choice set, the definition of a social choice function (SCF) over the subset  $A$  may be given as a functional relation that defines a non-empty choice set for every non-empty subset of  $A$ .

A final definition which must be considered is the property of "foundedness".<sup>36</sup> Foundedness is a condition where for any subset  $A$  of  $S$  there does not exist an infinitely long descending

chain of the type  $(\dots x_3 R x_2 \& x_2 R x_1)$  so that the alternatives  $x$  are infinite. Using the above definitions, two analysts, Sen and Pattanaik<sup>37</sup> were able to prove the following theorem: a necessary and sufficient condition for  $R$  to generate a non-empty maximal set for every non-empty subset of  $S$  is that  $P$  should be founded over  $S$ . If one notes that the binary relations,  $R$  and  $P$ , are also reflexive and connected, then an additional theorem may be derived:<sup>38</sup>  $R$  generates a choice function over  $S$  if and only if  $R$  is reflexive and connected and  $P$  is founded over  $S$ .

Next the above analysis will be applied directly to the spatial model. Take any dimension in the spatial model which possesses the properties outlined in Section 2. Let a constraint constant be introduced into the model so that the dimension is partitioned into two intervals:  $A = (x, \rightarrow)$  and  $B = (\leftarrow, x]$ , where the constraints constant  $b$  is given as being less than or equal to  $x$ . In effect,  $B$  will be eliminated from the set of feasible alternatives by stipulation. Graphically, the following would result, where the shaded area represents  $B$ .

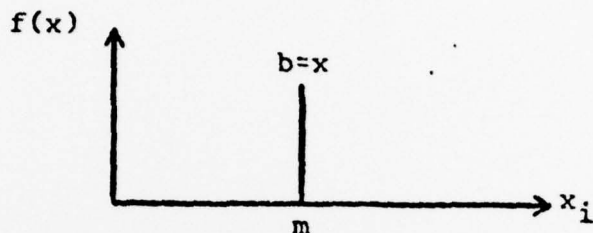


Since by construction,  $A$  has as one of its interval end-points an open-ended element  $x$ , such that for any point chosen in  $A$  there exists another point which is more preferred, no equilibrium point exists for the individual.

Since there exists the possibility that no dominant position exists over a given subset of alternatives ordered by an individual, it remains to be seen what effect this engenders in the social choice.

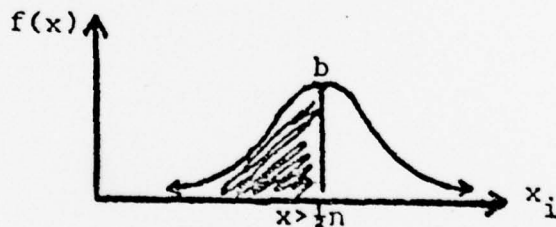
Since the constraint constant affects all citizens, some citizens or no citizen, several cases concerning the constant must be discussed.

Case one: consider a situation in which all citizens prefer the same alternative as their best choice. Let the constraint constant  $b$  for each citizen equal the constraint over the social choice preference ordering so that  $b$  equals the median of the density function  $f(x)$ . Graphically, this may be represented as follows:



Clearly, under the case of unanimity above, no social choice is engendered.

Case two: consider a situation in which a majority of citizens prefer as their best choice, alternatives which are equal to or less than the constraint constant  $b$ , so that the set of alternatives desired by the majority are infeasible. Graphically, this may be represented as follows:

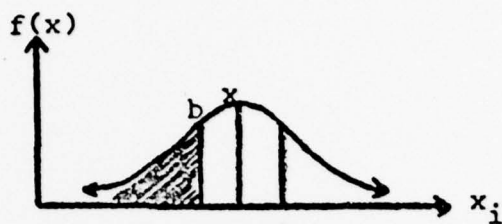


In the above case, the majority, assuming that everyone votes, must prefer an alternative which is greater than  $b$ , but less than any other point. If  $b = x_{n+1}$  and descending from that point the alternatives are given as  $x_n, \dots, x_2, x_1$ , then the alternatives in the feasible subset of  $S$  above are not founded. This is true since no matter what value is substituted for  $n$ , there exists at least one alternative which is more preferred.  $C(A)$  is therefore empty and no dominant point exists.

Case three: consider a situation in which a constraint constant is introduced such that the most preferred alternative for society is not eliminated as a feasible alternative, where



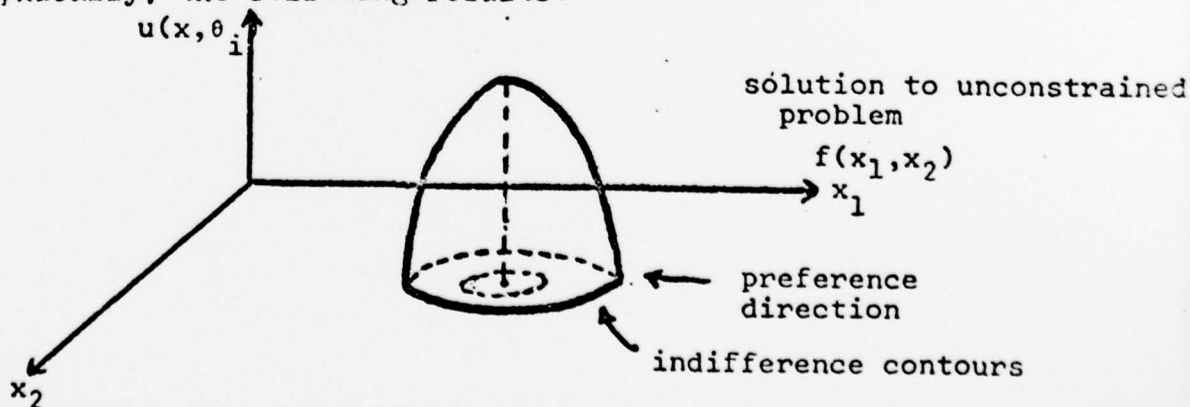
$b$  is less than the median. Again, all citizens are assumed to vote. Graphically this may be represented as follows:



In the above representation, it is clear that citizens, whose most preferred position is less than  $b$ , engenders a situation in which the most preferred point desired by society is not the equilibrium point, but the  $\bar{x}$  of the distribution, where  $\bar{x} \neq M$ . Also since the only restriction is that the constraint is less than  $b$ , there exists the possibility that the equilibrium point may not be unique.

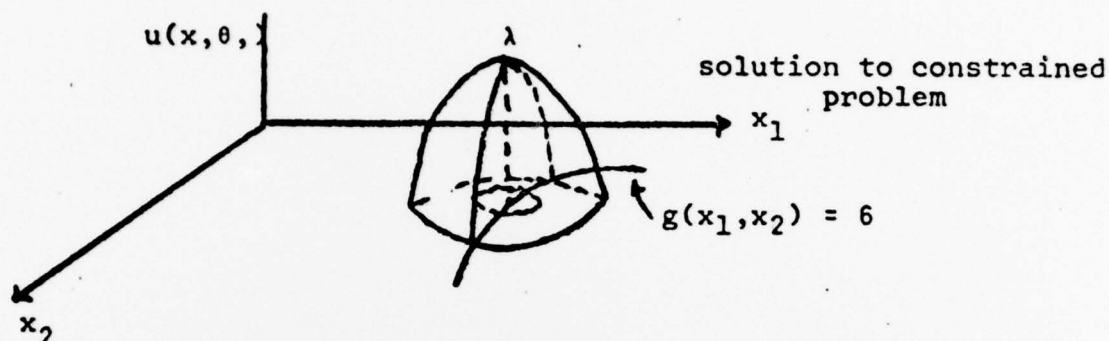
Case four: it appears that under a world of constraints, an equilibrium point only guaranteed, when for any given citizen, the constraint constant  $b$  is always unequal to  $x$ . This allows for the complete range of alternatives to be ordered over the density function  $f(x)$ .

From the above analysis, the study may now proceed to a consideration of a world of two or more dimensions, in order to discover the existence of an equilibrium point under the existence of constraints. Consider initially a world of two dimensions  $x_1$  and  $x_2$  which order an individual's preference profile into a utility function with a most preferred position  $h$ . Graphically, the following results:



Next, let a constraint function be introduced into the analysis so that the set of feasible alternatives is limited

by a vector in two dimensions,  $g(x) = b$ , where  $g_1(x_1, x_2) = b_1$  and  $g_2(x_1, x_2) = b_2$  such that  $b = (b_1, b_2)$ ; and where  $b = \lambda$ . Also let  $\alpha_i$  be any indifference contour not constrained by  $b$ . By construction,  $\alpha_i$  must always be less than  $b$  and less than  $\lambda$ . Graphically, the following results:



In order to show that there does not exist a most preferred indifference contour for a citizen in two dimensions in the above analysis, let an indifference contour which is most preferred equal  $\alpha_{n+1}$  and let the set of indifference contours descending from that contour equal  $\alpha_n, \dots, \alpha_2, \alpha_1$ . As in the case of one dimension, no matter which contour is chosen, such that the points are elements of the real numbers, there always exists a contour which is more preferred.

It seems apparant that if the vector  $x = (x_1, x_2, \dots, x_n)$  is considered, as long as any element  $x_1$  to  $x_n$  contains as empty choice set in one or more dimensions, then as the number of constrained dimensions increases the greater the dispersion of possibilities about a social equilibrium point.

#### Several theoretical implications.

For those schools of thought which may advocate a "narrow thesis of empiricism," that is, those who deny either the importance, necessity or existence of theoretical terms,<sup>38</sup> the analysis might well have ended above. In so doing, the empirical and formal components in a dimensional analysis would remain the major determining factors. In the following sections, theoretical considerations will be introduced into the analysis of dimensions as a concept of measurement in order to show that theoretical concerns are of considerable importance to

dimensional analysis.

If a dimensional model is to be of any value theoretically, it must be presented so that its relevance and relationship to other models, concepts and constructs within a theoretical paradigm and perhaps between theoretical paradigms is clearly established. Take for example the dimensional equation for centripetal force<sup>39</sup> given as  $f = \phi(m, v, r)$ , where  $f$  is centripetal force,  $\phi$  is a function defined over mass  $m$ , velocity  $v$ , and radius  $r$ . From the analysis thus far, the designation of primary and secondary dimensions is entirely arbitrary. One set of dimensions which might be used would be  $m = m$ ,  $v = lt^{-1}$ , and  $r = l$  so that the equation for force becomes  $f = k mv^2/r$ , where  $k$  is a dimensional constant. Suppose that for some reason our analysis posits the same dimensions for time and length. Then the analysis would derive  $f = ml^{-2}$ ,  $v = l^0$ ,  $r = l$  and it can be seen that any one of an infinite number of combinations  $m/r$ ,  $mv/r$ ,  $mv^2/r$ ,  $mv^n/r$  would satisfy the equation. Since dimensional analysis seeks a specific solution, this solution seems to suggest that the above formula becomes somewhat meaningless for much of conventional analysis in physics so that new concepts of measurement must be developed for the entire system. From the above, then, one might conclude that theoretical import gives rise to the inclusion and/or exclusion of certain kinds of dimensions in various combinations according to the theoretical and observational nature of the quantities being analyzed. The combinations decided upon in turn determine the kind of theoretical explanation which may be offered and which may not. It, therefore, becomes important for the dimensional equation to be constructed so that its relevance to theoretical enterprises or paradigms is clearly established. If this is not accomplished, then it will be difficult or impossible to tell from which theory a model is derived or how it may be included in any theory. Ultimately, its contribution to scientific knowledge cannot be assessed, that is, its contribution toward postdiction, prediction and explanation will be unclear.



Spatial modelling: is it properly included within Arrow's paradigm?

Apparently, the spatial modelling enterprise derives at least partially from the work of Anthony Downs, but the main derivation appears to stem from Arrow's work as characterized in the introduction. It might be useful to look at both formulations to discover whether or not the spatial model interpretation may be considered to be within the same theoretical framework as that of Arrow. Taking the two approaches point by point, the following areas of divergence may be elucidated.

Initially and emphatically, Arrow prohibits the use of utility functions of any kind. This is perhaps most clearly illustrated by Arrow's Condition 3, The Independence of Irrelevant Alternatives, which explicitly eliminates utility functions, but is not used directly in the proof of the General Impossibility Theorem.<sup>40</sup> Arrow's reasoning in this regard may be summarized as follows:<sup>41</sup> (1) they are not measurable for one individual, (2) they cannot be compared across individuals, (3) there are an infinite number of possible expressions for utility in terms of functions so that choosing any one is essentially a normative judgment, and (4) they are unnecessarily restrictive with regard to additional and alternate assumptions. The spatial model, of course, assumes that the possibility of expressing individual preferences as utility exists. Although the approach admits of the possibility of an infinite variety of functions, they limit their analysis to the class of functions listed in the explication of the spatial model. The approach also considers possible restrictions or assumptions, some of the most important of which are: (1) functional forms of utility functions for each individual are identical, (2) individuals weight dimensions in an identical fashion, (3) individuals assign the same degree of relative importance to all issues vis-a-vis one another.

Another topic wherein divergence is high between the two approaches would be with regard to the specific assumptions made about alternatives for social choice. In Arrow's formulation



of the problem of rational social choice, he explicitly requires that alternatives be discrete over the entire set of alternatives, as well as over any subset.<sup>42</sup> Further, he requires that the set of alternatives for social choice be finite.<sup>43</sup> Apparently, Arrow was aware of potential problems involved in dealing with continuity and infinity. The spatial model approach assumes quite the opposite, that is the spatial model works only when the alternatives ordered for an individual preference profile are assumed to be continuous as well as infinite.

Both approaches essentially require that certain limitation on the orderings of alternatives for social choice. In other words, they both define those orderings which will be admissible. For Arrow, orderings are admissible if they satisfy Axioms 1 and 2 and his five conditions. Later in his work, Arrow relaxes Condition 1 in order to admit orderings which are only single-peaked.<sup>44</sup> This causes the problem not to be cast in terms of social welfare functions any longer, but does provide a sufficient condition for eliminating the Impossibility Theorem. Other analysts clearly in the tradition: Sen, Pattanaik and Inada have added necessary conditions as well, merely by relaxing Condition 1. The spatial approach also assumes the property of single-peakedness, but they also add in utility maximization thereby violating Condition 3 of Arrow's work. The difference in the two approaches is clear: in the former, the relaxation of a minimal number of conditions is paramount, while in the latter, the concern is not with retaining conditions and attaining solutions within the social welfare function framework.

One of the most important characteristics of Arrow's analysis is that it applies to all decision rules meeting Arrow's five conditions such that it is completely general.<sup>45</sup> This means that any attempt to discover a specific rule which would avoid the Impossibility Theorem will not succeed unless the axioms and conditions are changed. The spatial model does not possess this characteristic, however, since it applies only to methods of majority decision-making which are compatible to a dimensional interpretation.

Yet another significant point of divergence between the two approaches would be the question of the possibility of a dimensional solution; this of course has been the central theme of this paper. Clearly, Arrow's formulation of the problem is entirely non-dimensional. Therefore, it is not subject to the kinds of criticism presented above which derive from one dimensional assumption or another. In addition to not being subject to the criticisms above, Arrow's approach is not affected by the problem of accounting for a fixed structure which relates to a phenomenon which is highly variable over time.<sup>46</sup> Arrow's Conditions 2 and 3 account for preference orderings at any one point in time. The spatial model, however, is a fixed multidimensional structure which may deal with a highly variable phenomenon. Therefore, it would seem that the spatial model is highly restricted in that it cannot account for dimensions which are highly variable, temporary or possibly irrelevant.

A final point of divergence which encompasses all of the above criteria is that of the number of essential assumptions and restrictions required by each model. For Arrow and indeed rational choice theorists clearly in this tradition, the problem of social choice seems to be determined by positing an absolute minimum number of restrictions upon decision-making situations.<sup>47</sup> The spatial model as evidenced throughout this analysis requires a good deal more in the way of assumptions, and therefore, may be seen to be considerably more restricted and consequently highly limited in possibilities for application.

Given the above points of divergence, the possibility for theoretical commonality between the two may be discovered by examining the nature of the results which Arrow is trying to achieve and the results of the spatial analysts in comparison with group decision rules.<sup>47</sup> To begin with, group decision rules in general may be shown to be distinguishable into subsets of one another according to the degree of restrictiveness imposed by the conditions characterizing each. The set of rules in which all others are contained is simply labeled a rule. A rule may be defined as a functional relation of the range of which

constitutes a set of binary weak preference relations defined over  $S$  and the domain of which is a class of ordered sets of binary weak preference relations defined over the set of all alternatives  $S$ . The notation for a rule is  $R = f(R_1, \dots, R_n)$ . Contained within the set of rules is the subset dealing only with social choice. These rules are called group decision rules. The conditions imposed upon these two kinds of rules are not stringent. For example, relations may be connected, reflexive or transitive, but they need not all occur together.

A more restricted group decision rule which requires at least two necessary conditions, reflexivity and connectedness, is the social choice function (SCF), defined briefly as a group decision rule which defines over every non-empty subset of  $S$  a non-empty choice set,  $C(A)$ . Stated in less technical terms, a social choice function exists when for any subset of alternatives in  $S$ , there exists a unique alternative which is as good as or better than any other alternative in the subset. Along with the two necessary conditions above, several other conditions may be added in various combinations in order to guarantee a non-empty choice set.

The social choice function may be further narrowed by requiring it to be a social decision function (SDF). A social decision function may be defined as a social choice function such that every social weak preference relation in its range engenders a social choice function over  $S$ . A social decision function which has as its range a set of complete social orderings—that is, orderings which are reflexive, connected and transitive—characterizes a social welfare function (SWF).

Initially, it must be noted that the beginning definition for a rule also is a subset of other less restrictive mathematical relations, while a social welfare function is highly restricted, represents only some possible restrictive conditions, but certainly not all. The above explanation may be represented more lucidly as follows:

rules  $\subset$  GDR  $\subset$  SCF  $\subset$  SDF  $\subset$  SWF  $\subset$  other rules

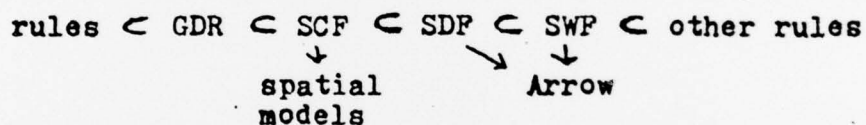
Arrow's theoretical world assumes as a minimum, the conditions necessary to guarantee a social welfare function. Of course, as



mentioned earlier, these conditions may lead to intransitive social preference profiles and dictatorial choices for society. The imposition of single-peakedness on Arrow's condition 1 eliminates both problems above while only restricting the range of feasible alternative orderings.

Spatial theorizing is not concerned so much with retaining Arrow's conditions as it is with representing the necessary and/or sufficient conditions for an equilibrium point to exist under majority rule in more than one dimension under the assumption that individual ordinal utility functions will lead to the generation of such a point if in fact one exists. In terms of the functions delineated above, spatial modeling seeks only a social choice function with its own set of restrictive conditions.

The initial representation of the sets of rules may be modified to show Arrow's formulation and the spatial model:



From this it is clear that with the utilization of utility functions, multidimensionality and other properties and conditions the spatial model approach cannot move up the latter of restrictive subsets to Arrow's SWF and beyond to other more restrictive rule, since this has become impossible by definition. This does not of course imply that the SCF subset of the latter of restrictiveness is somehow inferior to the SDF or SWF. It does, however, imply that both approaches being considered above are not theoretically in the same tradition.

#### An additional theoretical implication.

An important consequence of viewing quantities and numerical laws as being independent of the dimensions which may be used to express them is that the possibility for theoretical deductive analysis and manipulation may proceed even though empirical means for observation and dimensional measurement have not been developed. One good example of this would be the development of the theory of relativity which links Newton's laws of motion with the laws of motion for light rays. Generally, the theory



contains constructs which are as of yet unmeasurable or unobservable, but nevertheless it can explain both sets of phenomenon in a "unified" way better than any other competitor.<sup>48</sup> This important aspect of deductive theory seems to be most significant for doing science when the quantities and laws are interpreted; since if they are not interpreted, the results of any deductive manipulations will be of interest only to the mathematician or logician.

#### IV. Summary and conclusions.

In the above analysis, three concepts of dimension applicable to social science were considered: the ordinary language concept, geometrical concept, and dimension as a concept of measurement. Each concept was shown to have important consequences in the pursuit of scientific knowledge when compared and contrasted with the spatial modelling enterprise.

With regard to formal properties of quantities, numerical laws and dimensions, it was shown that quantities and laws determine dimensional models, but that working backwards from dimensional models in order to deduce unknown quantities and laws was not generally possible except perhaps in some fortuitous manner. This lead to the conclusion that dimensional analysis is not an a priori means for doing science, but instead, "an analysis of an analysis" which has already been completed.<sup>49</sup> It was also shown that according to this interpretation there exists a possibility of three kinds of solutions: contradictory, irrelevant and indeterminant. This possibility indicated the necessity of specifying interpreted dimensions so that a given solution can be evaluated as a solution to a specific problem. The analysis of the spatial model in these terms suggested that the model could not account for the consequences and implications of dimensions as a concept of measurement.

Just as the formal antecedents of dimensional interpretations determine the formal structure of a dimensional model, the analysis also suggested that an empirical element must be considered. The empirical element was shown to be a necessary element in a dimensional model and the notion further supported

the impossibility of an a priori interpretation and the possibility of alternate solutions. Again when compared to the spatial model, the empirical section of the analysis suggested that the spatial model could not adequately account for problems arising in this area.

By combining conclusions derived from the formal and empirical sections of the analysis, the following conclusion was drawn: if the spatial model by some fortuitous circumstances can in fact account for the formal and empirical criticisms rendered, it still appears to be at least an extra step in gaining knowledge about a scientific interpretation of a phenomenon.

Next, the analysis attempted to show that the spatial modelling enterprise appeared to be seeking a solution or solutions to the problem of rational social choice in a manner very different from the traditional works in the field. Given that the spatial modelling enterprise was not really seeking solutions to problems in the traditional formulation of rational social choice, it was suggested that the uninterpreted nature of the model was such that the relationship with other theoretical paradigms could not be established with the model in this form. Also closely related to this point is that the mathematical model does not specify what will and will not qualify as observation terms according to a theoretical framework. This again indicated that the model was somewhat unclear as to what it could provide solutions for with respect to specific phenomenon.

From the above summary, at least two very general conclusions might be drawn. First, perhaps when doing science, mathematics and mathematical structures should be viewed as means toward achieving explanation of a phenomenon and not as ends in themselves. This, of course, is not to say that it is improper to study mathematics as an end, since this is precisely what is done in the discipline of mathematics; instead, it is improper to study mathematics as an end in itself when doing science. And second, perhaps the problems of infinity and continuity, geometrical concepts of dimension and so on, indicate the need to develop new or alternate mathematical enterprises which are or

could be more conducive to social science explanation.

#### Footnotes

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